INTERPOLATION AND SMOOTHING OF EXPERIMENTAL DATA WITH SLIDING POLYNOMIALS

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ABSTRACT

This paper presents a comprehensive overview of the analytical structure of sliding polynomials. Emphasis is placed on explanation so that researchers with limited background in numerical analysis can apply the methods. Presentation is generalized to assist readers from various disciplines in preparing data for interpretation. Polynomials of increasing complexity were previously presented in separate publications. The intent here is to develop an appreciation of the common elements underlying three levels of complexity, starting with simple linear forms. Continuity of the first derivative of interpolation arcs is added first, followed by extension to continuity of the second derivative. The three forms are then transformed to smoothing operators using least squares.

INTRODUCTION

Interpolation and smoothing are routinely required to reduce and process experimental data. Choice of a particular method depends on the amount of data and on awareness of available methods. DuChateau et al. (1972) and Kimball (1974) dealt with interpolation and smoothing techniques that require substantial amounts of experimental data. Such data may be generated either in an extensive cyclic time domain or by repetition in a limited time or space domain. Savitzky and Golay (1964) discussed smoothing and differentiation of data using convoluting functions in weighted moving average techniques. Again, availability of substantial amounts of data is implied, generated by basically continuous physical experimentation. The author (1961, 1962, 1967a) showed how sliding polynomials could be used for interpolation and smoothing. This technique can be used with more limited amounts of experimental data.

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This publication draws together and expands the author's previous work on sliding polynomials. The construction of sliding polynomials is reviewed and analyzed. Common elements and concepts can be found in a hierarchy of polynomials. Tracing these common elements through three levels of complexity will be useful to researchers in many disciplines who must process observational or experimental data. Realization of the full potential of the sliding polynomials will aid in choosing interpolation and smoothing techniques best suited to specific data processing requirements.

Understanding the hierarchial structure of the polynomials can begin with review of simple linear interpolation. Extension of this simple form to piecewise linear functions establishes a basis for understanding more complex forms. Addition of a requirement for mathematical continuity of a first derivative is a logical next step. Further extension follows with requirement for mathematical continuity of a second derivative. All of these three piecewise forms are utilitarian. While a basic knowledge of mathematical and numerical analysis is assumed

phasis is placed on explanation to aid the reader in selection of the form suited to his purposes.

INTERPOLATION WITH POLYNOMIALS

Interpolation is the estimation of values between given discrete values of a property. Some function must be assumed. This function must give values of the property dependent upon values and locations of the given discrete points. A one-to-one correspondence exists between mathematical degrees of freedom in the function and the number of given discrete points. Data points through which the function is required to pass will be called base points.

In the following discussion, basic polynomial properties and the functional forms of sliding polynomials are explained. Then, the use of these concepts in interpolation is described.

Basic Polynomial Properties

Polynomials of any degree could be assumed for interpolation. Beyond the fifth power, however, the algebraic manipulations become tedious and error prone. A fifth-power polynomial, evaluated from six or more coordinate points, is sufficient to produce smoothly continuous rates of change through experimental data. Therefore, this presentation does not go beyond fifth-power polynomials.

The fifth-power polynomial has the conventional form

$$y=a+bx+cx^2+dx^3+ex^4+fx^5 \tag{1}$$

Coefficients a through f represent six mathematical degrees of freedom which can be satisfied by reference to six given discrete points. These concepts are developed under "Six-Point Interpolation."

The first derivative of equation 1, the rate of change of y, is

$$dy/dx=b+2cx+3dx^2+4ex^3+5fx^4. \tag{2}$$

Higher order derivatives are given in equations 3 through 6.

$$d^2y/dx^2=2c+6dx+12ex^2+20fx^3. \tag{3}$$
$$d^3y/dx^3=6d+24ex+60fx^2 \tag{4}$$
$$d^4y/dx^4=24e+120fx. \tag{5}$$
$$d^5y/dx^5=120f. \tag{6}$$

Note the fundamental continuity of form in equations 1 through 5. Any one of these can be used as an interpolation function. Properties of the function for estimating rates of change are given by derivatives of order higher than the chosen functional form. Properties of the function for integration are given by derivatives of order lower than the chosen functional form. For example, equation 5, the straight line, has a constant rate of change given by equation 6 and a parabolic rate of accumulation given by equation 4. In the discussions of two-point, four-point, and six-point interpolation, equations 5, 3, and 1 are used to develop special polynomials. They act as flexible splines that pass through extended numbers of coordinate points. Since symmetry is required, only polynomials with even degrees of freedom can be used in this method.

Sliding Polynomials

Different terminology may be used to describe the mathematical techniques of passing a continuously transhaping function from left to right through some set of discrete values, or base points. As the function moves to the right, base points must drop out on the left side of the function span, and new points must be picked up on the right side of the function span. This concept can be called continuous transforming, flexible spline, or sliding polynomial. The term "sliding polynomial" is considered most suitable in the development that follows.

First, consider the schematics in figure 1. They are sufficient for a simple situation where interpolation is always within one interval between base points.

Note the set of coordinate points $YB0$ through $YB6$ in figure 1a. These coordinate points, spaced at uniform intervals, will be used as base points through which interpolation polynomials are required to pass. The designation $YB0$, for example, means a base point in y at a value of x equal to zero. Uniform spacing of the base points is not absolutely necessary, but it does greatly simplify the algebraic forms.

Now observe in detail the interval between $YB2$ and $YB3$. Figure 1b is the schematic representation for passing a straight line between the two points. Interpolation can be made on this straight line. Figure 1c shows a cubic equation passing through $YB1$, $YB2$, $YB3$, and

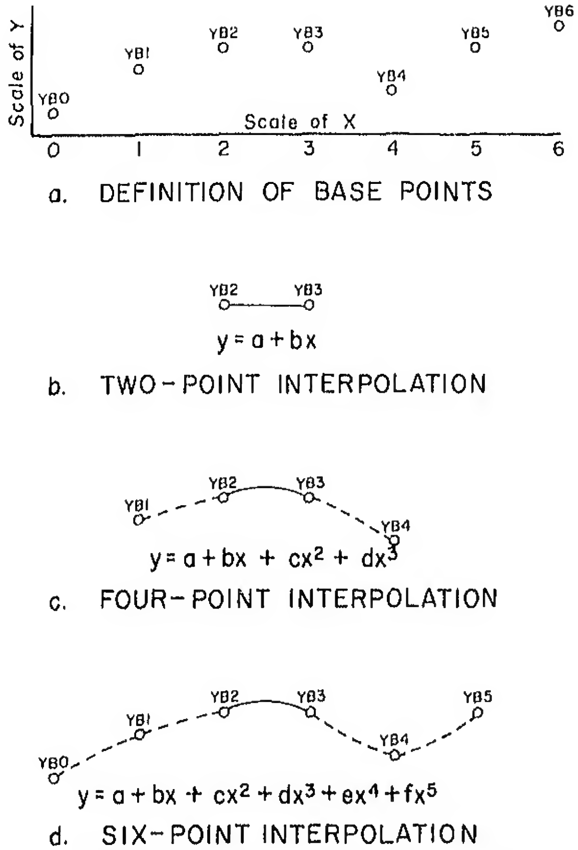

FIGURE 1 —Definition of interpolation systems.

*YB*4. Interpolation is to be limited to the interval between *YB*2, and *YB*3, the same interval covered by the straight-line example. Figure 1d shows a fifth-power equation passing through six points. Interpolation again is to be limited to the interval between *YB*2 and *YB*3.

A more realistic, and more complicated, problem is interpolating in adjacent intervals (fig. 2). Figure 2a shows interpolation on two adjacent linear segments. Visualize a point of interpolation on the x-scale moving from *YB*2 to *YB*3. Values of y can be read from the straight line connecting *YB*2 and *YB*3. When the point passes through *YB*3, a transformation is necessary. Point *YB*2 is dropped on the left, and point *YB*4 is added on the right. As the x-point moves through $x=3$, interpolation changes abruptly from the line through *YB*2 and *YB*3 to the line through *YB*3 and *YB*4.

A smoother transition through *YB*3, utilizing an arc through *YB*2 and *YB*3 and one through *YB*3 and *YB*4, is shown in figure 2b. The technique to suppress the angle between linear segments at *YB*3 is developed under "Four-Point Interpolation." It is based on dropping out *YB*1 on the left and picking up *YB*5 on the right. The phasing out and phasing in of the base points is gradual, instead of abrupt, as in the linear segments.

Figure 2c is an extension of concepts in figure 2b. The higher order polynomial phases out *YB*0 and phases in *YB*6 as the x-point leaves the *YB*2-*YB*3 arc and passes through $x=3$ into the *YB*3-*YB*4 arc. Special forms for this method are shown under "Six-Point Interpolation."

Two-Point Interpolation

Two-point interpolation is the familiar and conventional linear interpolation. The method is intuitively understood and used by many persons. This intuitive understanding is used here as the basis for presentation of more complex polynominal methods. To prepare for this presentation, the familiar linear interpolation is cast into algebraic forms which will be needed later. Only two base points at a time are used in linear interpolation. In figure 1a, the two points are *YB*2 and *YB*3. The two parameters in the associated equation can be evaluated in terms of the base points. The matching of number of parameters and number of base points should be noted.

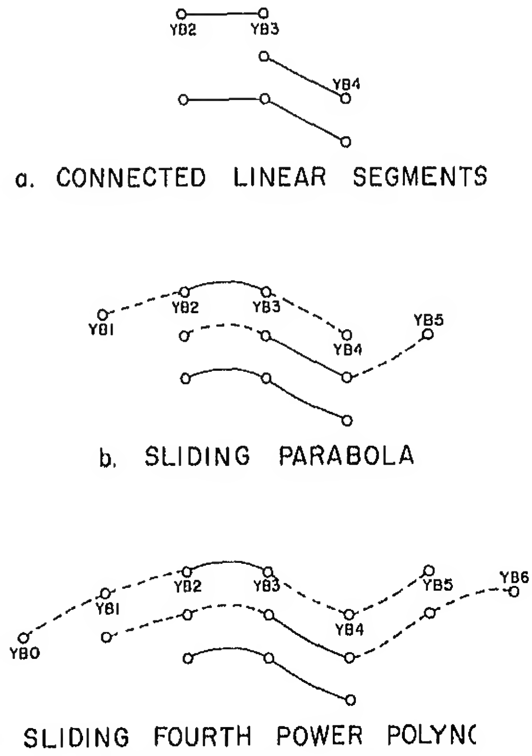

FIGURE 2.—Definitions for connected arcs.

Define a new scale variable z, with $z=0$ at $x=2$, and $z=1$ at $x=3$. An equation to define the assumed linear interpolation function between $YB2$ and $YB3$ can now be written:

$$y=a+bz. \tag{7}$$

At $z=0$, $y=YB2$, and $z=1$, $y=YB3$. Substituting these coordinate values of the two base points in equation 7 yields the simultaneous equations

$$YB2=a \tag{8}$$

$$YB3=a+b \tag{9}$$

From equations 8 and 9, $a=YB2$, and $b=YB3-YB2$. Equation 7 may now be written in terms of the base points as

$$y=YB2+(YB3-YB2)z=(1-z)YB2+z\cdot YB3. \tag{10}$$

The solution for equations 8 and 9 may also be written in the form of the matrix-vector equation

$$\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} \cdot \begin{vmatrix} YB2 \\ YB3 \end{vmatrix} = \begin{vmatrix} a \\ b \end{vmatrix} . \tag{11}$$

So-called interpolation coefficients can be developed from special values of $YB2$ and $YB3$. First, set $YB2$ equal to 1 and $YB3$ equal to zero. Expansion of equation 11 gives $a=1$ and $b=-1$. The special form of equation 10 results in

$$y_a=1-z. \tag{12}$$

Since $YB2$ was set to unity, y_a can be called the unit contribution of a base point at $z=0$, the left one of two base points. Next, set $YB2$ equal to zero and $YB3$ equal to 1. Expansion of equation 11 gives $a=0$ and $b=1$. The special form of equation 10 now is

$$y_b=z. \tag{13}$$

Now, y_b can be called the unit contribution of a base point at $z=1$, the right one of the two base points.

Equations 12 and 13 can be evaluated for values of z as in table 1. The values of y_a and y_b are normally designated as $C0$ and $C1$, meaning the interpolating coefficient to be multiplied by the base value at $z=0$ and by the base value at [illegible] meaning is expressed on the [illegible] 0. For example, two-point [illegible] -fourth of the x-dis- [illegible] ritten directly, with [illegible] 1, as

$$y=(0.75)YB2+(0.25)YB3. \tag{14}$$

Equation 14, of course, is the familiar and conventional form for linear interpolation.

If the two adjacent linear segments in figure 2a are considered, equations can be written directly for interpolating y_1 on the left segment and y_2 on the right segment:

$$y=YB2+(YB3-YB2)z, \tag{15}$$

and

$$y=YB3+(YB4-YB3)z. \tag{16}$$

With $z=1$, equation 15 gives $y_1=YB3$. Also, with $z=0$, equation 16 gives $y_2=YB3$. This simply confirms that both segments pass through $YB3$, and the interpolation is continuous in the sense that there is no gap at the base point $YB3$. However, $dy_1/dz=YB3-YB2$ and $dy_2/dz=YB4-YB3$ show that the slopes of the segments are different, constant, and not continuous.

TABLE 1—*Two-point interpolation coefficients*

z	y_a	y_b
0.10	0.90	0.10
.25	.75	.25
.50	.50	.50
.75	.25	.75
.90	.10	.90

Four-Point Interpolation

In geometrical concept, two-point interpolation can be visualized as a point being linearly displaced in y-scale as it moves along the z-scale from one base point to the next. This two-directional displacement generates a locus of interpolated values which lie along a straight line. Note particularly the concept of a point in continuous displacement generating a straight-line interpolation locus, and note that the line is one mathematical order up from the point.

In four-point interpolation, the geometrical concept is a function that will be linearly transhaped in y-scale as a point of interpolation moves along the z-scale. This function is up two orders from the point, and is, therefore, a parabola. The locus of interpolated values is one order higher than the function being transhaped and is a cubic polynomial.

The geometrical concept on which four-point interpolation is based is further developed as follows: Visualize a parabola that passes through base points $YB1$, $YB2$, and $YB3$ in figure 1. Next, visualize the parabola through $YB2$, $YB3$, and $YB4$. As the point of interpolation moves along

the x-scale from $YB2$ to $YB3$, the interpolated value is to be read from a flexible parabola that transhapes linearly from the left parabola to the right parabola.

The analytical concept of four-point interpolation requires specification of four mathematical constraints on which solution for the four coefficients of the cubic polynomial, equation 3, are to be based. The mathematical specifications are based on commonality. The cubic equation is required to be common to the parabolas at the ends of the interpolation arc. This means simply that both the parabolas and the cubic pass through $YB2$ and $YB3$ (fig. 1c). In addition, the parabolas and the cubic are required to have the same slope, or the same first derivative, at $YB2$ and $YB3$.

Next, specify a scale variate z, which takes values 0, 1, 2, and 3 from left to right in any four-point set of base points. Define the left parabola by

$$ {}_1y=u+vz+wz^2 \tag{17}$$

and

$$ d_1y/dz=v+2wz. \tag{18}$$

Solve for coefficients u, v, and w by passing through the three points $z=0$, ${}_1y=YB1$; $z=1$, ${}_1y=YB2$; and $z=2$, ${}_1y=YB3$, giving the values

$$u=YB1, \tag{19}$$

$$v=-(3/2)YB1+2\ YB2-(1/2)YB3, \tag{20}$$

and

$$w=(1/2)YB1-YB2+(1/2)YB3 \tag{21}$$

Next define the right parabola by

$$ {}_2y=u+vz+wz \tag{22}$$

and

$$ d_2y/dz=v+2wz \tag{23}$$

Solve for new coefficients u, v, and w by passing the parabola through points $z=1$, ${}_2y=YB2$; $z=2$, ${}_2y=YB3$; and $z=3$, ${}_2y=YB4$. The solution is

$$u=3\cdot YB2-3\cdot YB3+YB4, \tag{24}$$

$$v=-(5/2)YB2+4YB3-(3/2)YB4, \tag{25}$$

and

$$w=(1/2)YB2-YB3+(1/2)YB4. \tag{26}$$

Define the cubic equation by

$$y=a+bz+cz+dz^3 \tag{27}$$

and

$$dy/dz=b+2cz+3dz^2 \tag{28}$$

The four mathematical constraints expressing commonality between parabolas and cubic can now be stated as four simultaneous equations (29-32). Equations 17 and 18, with coefficients in equations 19, 20, and 21, are set equal to equations 27 and 28 for $z=1$. Equations 22 and 23, with coefficients in equations 24, 25, and 26, are set equal to equations 27 and 28 for $z=2$. The resulting four simultaneous equations are

$$YB2=a+b+c+d, \tag{29}$$

$$(1/2)YB3-(1/2)YB1=b+2c+3d, \tag{30}$$

$$YB3=a+2b+4c+8d, \tag{31}$$

and

$$(1/2)YB4-(1/2)YB2=b+4c+12d. \tag{32}$$

The solution to these equations is

$$a=2\cdot YB1-3\cdot YB2+3\cdot YB3-YB4, \tag{33}$$

$$b=-4\cdot YB1+(19/2)YB2-8\cdot YB3+(5/2)YB4, \tag{34}$$

$$c=(5/2)YB1-7\cdot YB2+(13/2)YB3-2\cdot YB4, \tag{35}$$

and

$$d=-(1/2)YB1+(3/2)YB2-(3/2)YB3+(1/2)YB4. \tag{36}$$

This solution can be expressed concisely in the matrix-vector equation

$$(1/2)\begin{vmatrix} 4 & -6 & 6 & -2 \\ -8 & 19 & -16 & 5 \\ 5 & -14 & 13 & -4 \\ -1 & 3 & -3 & 1 \end{vmatrix} \begin{vmatrix} YB1 \\ YB2 \\ YB3 \\ YB4 \end{vmatrix} = \begin{vmatrix} a \\ b \\ c \\ d \end{vmatrix} \tag{37}$$

Next, consider the property of the locus of interpolated values at the common point of two arcs. Specifically, in figure 2b, arcs $YB2$-$YB3$ and $YB3$-$YB4$ obviously join at $YB3$. Now at $YB3$, interpolation arc $YB2$-$YB3$ ends on the parabola through $YB2$, $YB3$, and $YB4$. However, interpolation arc $YB3$-$YB4$ takes off from this same parabola. The four mathematical specifications require the interpolation cubics to be common and tangent to the parabolas at the ends of the arcs. The cubics of the $YB1$-$YB4$ set and the $YB2$-$YB5$ set are, therefore, both common and tangent to the same parabola at $YB3$. The cubics of adjacent interpolation arcs are thus common and tangent to each other.

Thus, it has been demonstrated that four-point interpolation based on a sliding, transhaping parabola generates values on a cubic polynomial. Adjacent cubic arcs are continuous and have continuous first derivatives as the interpolation locus passes through the base points.

Figure 3 is an example of four-point interpolation. (Computer program A-1, appendix A, generated the interpolated values.) Remember, when looking at figure 3, that each interpolation arc is a different cubic polynomial, with the ensemble of cubics having the unique property that they are common and tangent at the base points.

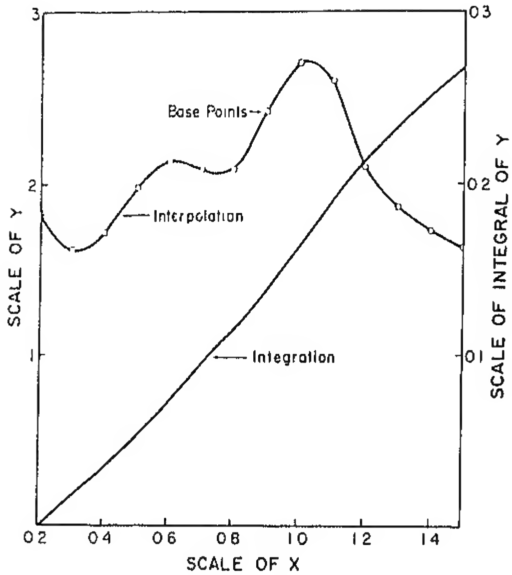

FIGURE 3 —Four-point interpolation and integration.

In summary, four-point interpolation establishes a form-free ensemble with two orders of continuity. Generally, the interpolation locus will pass through the base points without undesirable oscillation. In some situations, with an abrupt change in slope of the base point data, a tendency to oscillate may develop.

Four-point interpolation can be reduced to coefficients that operate directly on the base points, just as the two-point system was reduced. Consider the special case of $YB1=1$ and $YB2=YB3=YB4=0$. Then, from equations 33-36 or equation 37, $a=2$, $b=-4$, $c=5/2$, and $d=-1/2$, and the interpolation cubic is

$$y_0=2-4z+(5/2)z^2-(1/2)z^3. \tag{38}$$

Equation 38 can be evaluated for selected values of z, producing a coefficient that gives the unit contribution of the base point $YB1$ to the interpolated value. Since $YB1$ is at a z-value of zero, call this coefficient $C0$.

Now use the special case $YB2=1$ and $YB1=YB3=YB4=0$. The interpolation cubic from equations 33-36 is

$$y_1=3+(19/2)z-7z^2+(3/2)z^3. \tag{39}$$

Evaluate this for the same selected z-values used with equation 38. Call this set of coefficients $C1$, since $YB2$ is at a z-value of 1.

Similarly, evaluate $C2$ and $C3$. An interpolated value is the summation of the products of the unit operators with any real set of $YB1$ through $YB4$. This product summation is

$$y=y_0+y_1+y_2+y_3=C0\cdot YB1+C1\ YB2+C2\ YB3+C3\cdot YB4 \tag{40}$$

Table A-1 (appendix A) is a listing of the four interpolating coefficients for each one-hundredth of the interpolating arc from $z=1$ to $z=2$. These coefficients were generated by program A-2.

The area under any function is given by the integral of the function across a range of x (or z). The integral of the interpolation cubic of the four-point method is given in equation 41. Integration, just as interpolation, must be limited to the central interval of the four-point set:

$$\int_1^2 y dz=az+(b/2)z^2+(c/3)z^3+(d/4)z^4+C=A \tag{41}$$

The constant of integration, C, can be evaluated by noting that the area under the curve, A, is zero when $z=1$; that is, at the left end of the interpolation arc. For this value of z, and with equations 33-36, C is given by

$$C=(1/24)(-17\cdot YB1+5\cdot YB2-19\cdot YB3+7\cdot YB4). \tag{42}$$

The area under any segment of the interpolation arc can now be calculated with equations 33-36, 41, and 42. It is also possible to reduce four-point integration to a coefficient-multiplier scheme analagous to equation 40 for interpolations. Table A-2 is a listing of such coefficients, also generated by program A-2 (appendix A). Now, note that four-point integration is also included in program A-1. The area under the arcs of the interpolation cubic are plotted in figure 3. Remember that each arc starts with zero area. The total area to the left of any point is thus the sum of the partial area in the arc plus all full arcs to the left.

The coefficients in table A-2 are based on a scale variate z with the base points at unit intervals of z. If the base points are at a scale s in the original data, given by $s=kz+z_0$, then $ds=kdz$. An integral ydz can be changed to an integral $yds=kydz$ by simple change of variable. This change of variable is included in program A-1, but must be performed as a separate operation following use of table A-2.

Six-Point Interpolation

The development of the method of six-point interpolation has been given in detail previously (Snyder 1967a). The geometrical and analytical concepts are analagous to those used in development of the four-point method in the previous section.

Note briefly that six-point interpolation requires functions two orders higher than four-point interpolation, which in turn was two orders higher than the functions of two-point interpolation. Specifically, six-point interpolation is based on a sliding, transhaping, fourth-power polynomial, two orders up from the sliding parabola. Each arc of the interpolation locus is given by a fifth-power polynomial.

Six mathematical restraints are required. In addition to the four used for four-point interpolation, commonality of function and first derivative, commonality of the second derivative with the second derivative of the linearly transhaping, fourth-power polynomial is required at the ends of the interpolation arcs. The junctures of the arcs at the base points are mathematically continuous and have continuous first and second derivatives.

Details of the solution of the coefficients of the six-point method are not repeated here. The solution for the coefficients in equation 1 is conveniently expressed by the matrix-vector equation

$$(1/24)\begin{vmatrix} 432 & -2040 & 4080 & -4080 & 2040 & -408 \\ -918 & 4436 & -8752 & 8712 & -4346 & 868 \\ 765 & -3754 & 7414 & -7356 & 3661 & -730 \\ -313 & 1551 & -3078 & 3058 & -1521 & 303 \\ 63 & -314 & 626 & -624 & 311 & -62 \\ -5 & 25 & -50 & 50 & -25 & 5 \end{vmatrix} \begin{vmatrix} YB0 \\ YB1 \\ YB2 \\ YB3 \\ YB4 \\ YB5 \end{vmatrix} = \begin{vmatrix} a \\ b \\ c \\ d \\ e \\ f \end{vmatrix} \qquad (43)$$

In the previous section on four-point interpolation, it was pointed out that the method could be extended to integration of experimental data. An advantage of the six-point system is that it can be converted to differentiation, leading to determination of smooth rates of change of experimental data. The rate of change of values in equation 1 is given directly by equation 2. With coefficients b through f given by equation 43, rates of change can be computed by direct substitution in equation 2. An advantage of this method of interpolative differentiation is that the first derivatives are smoothly continuous because the second derivatives are continuous.

Figure 4 shows six-point interpolation through and the calculated first derivative of the same data as in figure 3. The interpolative values and derivatives were computed with program A-3 (appendix A).

Six-point interpolation and differentiation can be reduced to multiplying coefficients in a manner similar to interpolation and integration with coefficients as presented in the four-point method. Interpolation coefficients for each one-hundredth of the interval of the interpolation are from $z=2$ to $z=3$ are given in table A-3 (appendix A). These coefficients were generated by program A-4. Table A-4 gives coefficients for use in differentiation. Calculation of these coefficients is included in program A-4.

It should be noted that slopes given by the differentiation coefficients of table A-4 are in terms of the scale variate z. Given that the scale of original data, s, is related to z by $s=kz+z_0$, then $ds=kdz$. A slope dy/dz can be changed to a slope of original data by simple change of variable, yielding $dy/ds=(1/k)\,dy/dz$. This scale conversion is included in program A-3, but must

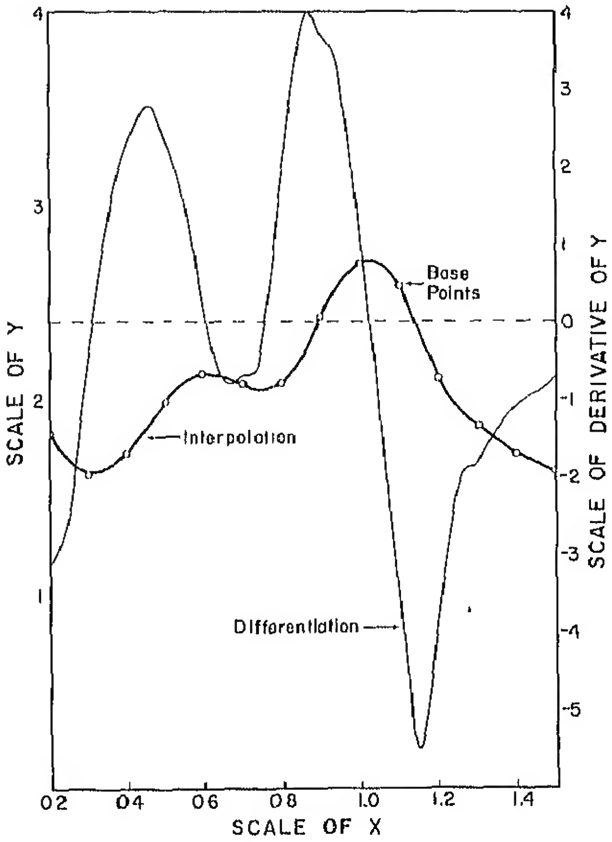

FIGURE 4.—Six-point interpolation and differ

be performed as a separate operation following use of table A-4.

SMOOTHING OF EXPERIMENTAL DATA

Smoothing of experimental data is essentially the application of the method of least squares to effect a transformation from recorded data to a set of base points for interpolation. Data may be recorded at nonuniform intervals. Transformation produces base points at uniform intervals. Data may include mensuration errors. These errors may be smoothed by averaging during the transformation. Successful averaging requires that base points be fewer in number than data points and also requires a configuration of data points conducive to production of good averages.

The location and spacing of the base points is a problem in design. No hard rules can be stated. The heart of the problem is the amount of averaging that is desired. With a goodly amount of data in a compact mass, a high degree of averaging can be used. The base points would likely be one-fifth to one-tenth of the number of data points. With limited data, strung out in range, the base points would likely be about one-half the number of data points, resulting in a limited degree of smoothing.

In the following examples, a high degree of smoothing will be evident. The placement of points is usually dictated by the particular problem. A base point would usually be placed at an interface or boundary, such as the surface of ground or water. In time series one point would usually be placed at time zero. Spacing of the remaining base points through the data range is obviously dictated by the degree of averaging and hence the number of points.

Two-Point Smoothing

Two-point smoothing is identical to evaluation of connected linear segments (Snyder 1967b). Equation 14 can be generalized as

$$y = C0 \cdot YB0 + C1 \cdot YB1. \tag{44}$$

In equation 44, the meaning is limited to one interpolation interval. If several adjacent intervals are considered, equations 45-47 result:

$$\begin{aligned} y_1 &= C0 \cdot YB0 + C1 \cdot YB1 && (45)\\ y_2 &= \qquad\qquad C0 \cdot YB1 + C1 \cdot YB2 && (46)\\ y_3 &= \qquad\qquad\qquad\qquad C0 \cdot YB2 + C1 \cdot YB3 && (47) \end{aligned}$$

In equation 45, y_1 is any value lying on the line from $YB0$ to $YB1$. In equation 46, y_2 is any value lying on the line from $YB1$ to $YB2$, and so on. The interpolation coefficients $C0$ and $C1$ are known functions of scale variate z, as in table 1.

Assume that experimental data consist of simultaneous coordinate values of x and y for a series of observations. Then, consider a series of base points to be substituted for the data points. It is necessary to specify the location of the base points within the x-scale of the data. Normally, one has some idea of the number of base points desired to span the data in the x-scale. Setting the location of the base points in x-scale automatically sets a z-scale value for each data point since the base points are always spaced at unit interval in z.

Referring now to equations 45-47, note that they are linear in form. Note also that given experimental values of y and given a computed z-variate for each observation, there is an equivalent of a series of observational equations as used in multiple regression. $C0$ and $C1$ take the position of x_0 and x_1 in regression, and $YB0$ and $YB1$ take the position of regression coefficients, conventionally designed b_0 and b_1. There is no regression intercept normally designated a. It is a simple matter, with conventional least-squares methods or programs, to compute $YB0$ and $YB1$ as regression coefficients. A transformation from experimental data in x-y coordinates to base point data, in z-$YB1$ coordinates, is thus accomplished. Note especially that in the configuration of equations 45-47, $YB1$ and $YB2$ must take values on two adjacent intervals. The continuity of the smoothing operation is thus satisfied with no gaps at the base points.

Table B-1 (appendix B) shows the details for organization of 30 data points into format for regression transformation. The data points and the result of least-squares transformation in two-point smoothing are plotted in figure 5. While the location of base points at x=0.4, 0.6, 0.8, and 1.0 is arbitrary, the values of these base points are wholly objective and the result of least-squares fitting. Note specifically in figure 5 the overall fit of the linear segments to the rough crescent shape of the experimental data. Since the two internal base points are each required to be common to two segments, the fit of a single segment to the data points in that segment is secondary.

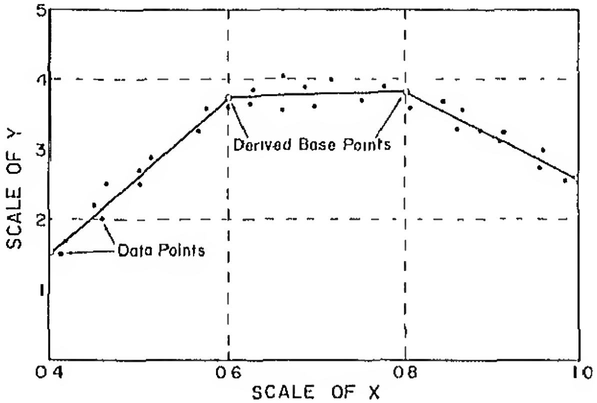

FIGURE 5 —Smoothing by two-point method.

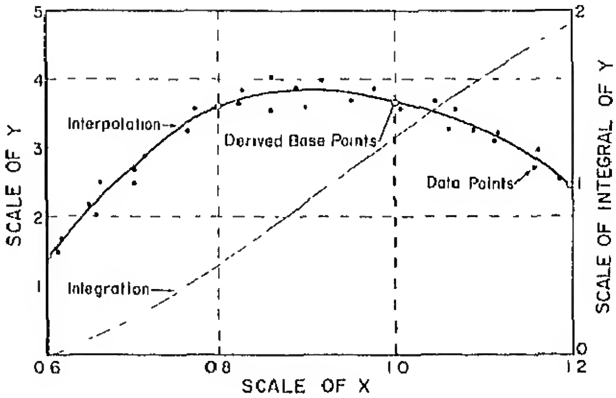

FIGURE 6 —Smoothing by four-point method.

Four-Point Smoothing

Four-point smoothing is a simple extension of two-point smoothing. Equation 40, modified for adjoining arcs, is extended in equations 48-51, analagous to the two-point equations, 45-47.

$$y_1 = C0 \cdot YB0 + C1 \cdot YB1 + C2 \cdot YB2 + C3\ YB3 \qquad (48)$$

$$y_2 = C0 \cdot YB2 + C1 \cdot YB2 + C2 \cdot YB3 + C3 \cdot YB4 \qquad (49)$$

$$y_3 = C0 \cdot YB2 + C1 \cdot YB3 + C2 \cdot YB4 + C3 \cdot YB5 \qquad (50)$$

$$y_4 = C0 \cdot YB3 + C1 \cdot YB4 + C2 \cdot YB5 + C3 \cdot YB6 \qquad (51)$$

Values for the coefficients are given in table A-1. Remember, in equations 45-47 each base point is linked to two adjacent intervals. $YB1$ and $YB2$ in those equations are each linked by $C0$ and $C1$. Note in equations 48-51 that an internal base point such as $YB3$ is linked to four interpolation arcs. The least-squares value of $YB3$ is dependent on experimental data in four intervals. Exterior base points are dependent on 1, 2, or 3 intervals.

Table B-2 (appendix B) shows organization of experimental data in format for four-point smoothing. The same data are used as for the two-point smoothing example, except the x-scale is shifted to the left to put the first base point at $x=0.4$. The z-scale is not affected. Program B-1 (appendix B) is used to perform the least-squares transformation. This program also generates the locus of interpolated values which runs through the derived base points and generates the intergrals of the interpolation arcs.

Figure 6 shows the results of the four-point smoothing. An interpolation locus of smoothly joined arcs is generated that runs continuously through the data. Comparison of figures 5 and 6 shows that the two-point and four-point methods produce base points not greatly different. The main difference is the creation of arcs by the four-point method, smoothly joined with angles eliminated. Values of the derived base points can be compared in table 2.

The integral of the interpolation arcs is also plotted in figure 6. This integral can be thought of as the smoothed summation of the experimental data.

Six-Point Smoothing

Six-point smoothing is a simple extension of methodology already presented. Equations 52-55 are analogous to equations 45-47 and 48-51.

$$y_1 = C0 \cdot YB0 + C1 \cdot YB1 + C2 \cdot YB2 + C3 \cdot YB3 + C4 \cdot YB4 + C5 \cdot YB5 \qquad (52)$$

$$y_2 = C0 \cdot YB1 + C1 \cdot YB2 + C2 \cdot YB3 + C3 \cdot YB4 + C4 \cdot YB5 + C5 \cdot YB6 \qquad (53)$$

$$y_3 = C0 \cdot YB2 + C1 \cdot YB3 + C2 \cdot YB4 + C3 \cdot YB5 + C4\ YB6 + C5 \cdot YB7 \qquad (54)$$

$$y_4 = C0 \cdot YB3 + C1 \cdot YB4 + C2\ YB5 + C3 \cdot YB6 + C4 \cdot YB7 + C5 \cdot YB8 \qquad (55)$$

In equations 52-55, note that an interior base point will depend upon values of experimental

TABLE 2. – *Comparison of derived base points*

Two Point		Four-Point		Six-Point	
x	YBi	x	YBi	x	YBi
				0.4	-4.14772
		0.4	-1.72742	.6	-1.55935
0.4	1.52418	.6	1.42889	.8	1.417502
.6	3.75141	.8	3.59534	1.0	3.59252
.8	3.81551	1.0	3.67674	1.2	3.68015
1.2	2.57954	1.2	2.44932	1.4	2.45502
		1.4	-.42235	1.6	-2.29737
				1.8	-20.17608

data in six adjacent intervals by linkage through the coefficients. Exterior base points may depend on five or fewer intervals of data.

Table B-3 (appendix B) shows organization of experimental data in format for six-point smoothing. Program B-2 is used to perform the least-squares transformation. Following transformation of the experimental data to a smoothed set of base points, program B-2 computes a locus of interpolated values and a locus of differentials of the interpolation arcs. Figure 7 is a plot of the experimental data and derived interpolation and differentiation loci.

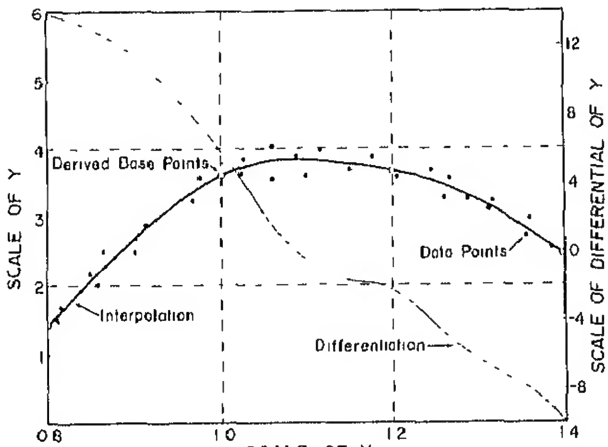

FIGURE 7.—Smoothing by six-point method

Figure 7 contains the same experimental data as figures 5 and 6. It can be seen that the interpolation arcs of the four-point and six-point methods are almost identical. Numerical comparison of the derived base values is shown in table 2.

Figure 7 also shows the differentiation of the interpolation arcs. This locus of values can be regarded as the smoothed rate of change of the experimental data. Note the smoothness of this locus through the base points and the near constant value of slope from about $x=1.14$ to $x=1.2$.

ATTRIBUTES OF SLIDING POLYNOMIALS

Figures 5, 6, and 7 demonstrate that various interpolation methods can be used to smooth experimental data. Smoothed values, whether read from two-point, four-point, or six-point interpolation loci, are not extremely different. Other methods would also produce acceptable interpolation loci. Kimball's (1974) method would require a means to separate the spectral portions of noise and information in the data.

The identifiable advantages of sliding polynomials are in the secondary characteristics of their loci. The four-point and six-point methods are smooth because their derivatives are continuous at the juncture points of the interpolation arcs. The four-point method, with a continuous first derivative, produces an extremely smooth locus of integral values, since integration itself is a smoothing process. The six-point method, with continuous first and second derivatives, can generate a smooth rate of change in experimental data. DuChateau et al. (1972) also used six points in evaluation. In their method, however, parabolic splines are required to be continuous only in the first derivative. Therefore, these splines cannot be expected to develop the smoothly continuous rates of change of experimental data. Such continuous smoothness is dependent upon a continuous second derivative.

Sliding polynomials and their integrals and derivatives can be derived from experimental data by conventional least-squares techniques. The interpolation coefficients provide exact error weights for each experimental data point, regardless of the interpolation interval in which the data point is located. The structure of equations 52-55, for example, shows that the evaluation procedure is similar to the use of smoothing convolutes as in Savitzky and Golay (1964).

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APPENDIX A.—COEFFICIENTS AND COMPUTER PROGRAMS FOR INTERPOLATION

TABLE A-1.—*Coefficients for four-point interpolation*

1.00	0.00000000	1.00000000	.00000000	0.00000000
1.01	-.00490050	.99975150	.00519850	-.00004950
1.02	-.00960400	.99901200	.01078800	-.00019600
1.03	-.01411350	.99779050	.01675950	-.00043650
1.04	-.01843200	.99609600	.02310400	-.00076800
1.05	-.02256250	.99393750	.02981250	-.00118750
1.06	-.02650800	.99132400	.03687600	-.00169200
1.07	-.03027150	.98826450	.04428550	-.00227850
1.08	-.03385600	.98476800	.05203200	-.00294400
1.09	-.03726450	.98084350	.06010650	-.00368550
1.10	-.04050000	.97650000	.06850000	-.00450000
1.11	-.04356550	.97174650	.07720350	-.00538450
1.12	-.04646400	.96659200	.08620800	-.00633600
1.13	-.04919850	.96104550	.09550450	-.00735150
1.14	-.05177200	.95511600	.10508400	-.00842800
1.15	-.05418750	.94881250	.11493750	-.00956250
1.16	-.05644800	.94214400	.12505600	-.01075200
1.17	-.05855650	.93511950	.13543050	-.01199350
1.18	-.06051600	.92774800	.14605200	-.01328400
1.19	-.06232950	.92003850	.15691150	-.01462050
1.20	-.06400000	.91200000	.16800000	-.01600000
1.21	-.06553050	.90364150	.17930850	-.01741950
1.22	-.06692400	.89497200	.19082800	-.01887600
1.23	-.06818350	.88600050	.20254950	-.02036650
1.24	-.06931200	.87673600	.21446400	-.02188800
1.25	-.07031250	.86718750	.22656250	-.02343750
1.26	-.07118800	.85736400	.23883600	-.02501200
1.27	-.07194150	.84727450	.25127550	-.02660850
1.28	-.07257600	.83692800	.26387200	-.02822400
1.29	-.07309450	.82633350	.27661650	-.02985550
1.30	-.07350000	.81550000	.28950000	-.03150000
1.31	-.07379550	.80443650	.30251350	-.03315450
1.32	-.07398400	.79315200	.31564800	-.03481600
1.33	-.07406850	.78165550	.32889450	-.03648150
1.34	-.07405200	.76995600	.34224400	-.03814800

TABLE A-1.—*Coefficients for four-point interpolation*—Continued

1.35	-.07393750	.75806250	.35568750	-.03981250
1.36	-.07372800	.74598400	.36921600	-.04147200
1.37	-.07342650	.73372950	.38282050	-.04312350
1.38	-.07303600	.72130800	.39649200	-.04476400
1.39	-.07255950	.70872850	.41022150	-.04639050
1.40	-.07200000	.69600000	.42400000	-.04800000
1.41	-.07136050	.68313150	.43781850	-.04958950
1.42	-.07064400	.67013200	.45166800	-.05115600
1.43	-.06985350	.65701050	.46553950	-.05269650
1.44	-.06899200	.64377600	.47942400	-.05420800
1.45	-.06806250	.63043750	.49331250	-.05568750
1.46	-.06706800	.61700400	.50719600	-.05713200
1.47	-.06601150	.60348450	.52106550	-.05853850
1.48	-.06489600	.58988800	.53491200	-.05990400
1.49	-.06372450	.57622350	.54872650	-.06122550
1.50	-.06250000	.56250000	.56250000	-.06250000
1.51	-.06122550	.54872650	.57622350	-.06372450
1.52	-.05990400	.53491200	.58988800	-.06489600
1.53	-.05853850	.52106550	.60348450	-.06601150
1.54	-.05713200	.50719600	.61700400	-.06706800
1.55	-.05568750	.49331250	.63043750	-.06806250
1.56	-.05420800	.47942400	.64377600	-.06899200
1.57	-.05269650	.46553950	.65701050	-.06985350
1.58	-.05115600	.45166800	.67013200	-.07064400
1.59	-.04958950	.43781850	.68313150	-.07136050
1.60	-.04800000	.42400000	.69600000	-.07200000
1.61	-.04639050	.41022150	.70872850	-.07255950
1.62	-.04476400	.39649200	.72130800	-.07303600
1.63	-.04312350	.38282050	.73372950	-.07342650
1.64	-.04147200	.36921600	.74598400	-.07372800
1.65	-.03981250	.35568750	.75806250	-.07393750
1.66	-.03814800	.34224400	.76995600	-.07405200
1.67	-.03648150	.32889450	.78165550	-.07406850
1.68	-.03481600	.31564800	.79315200	-.07398400
1.69	-.03315450	.30251350	.80443650	-.07379550

TABLE A-1.—*Coefficients for four-point interpolation*—Continued

1.70	-.03150000	.28950000	.81550000	-.07350000
1.71	-.02985550	.27661650	.82633350	-.07309450
1.72	-.02822400	.26387200	.83692800	-.07257600
1.73	-.02660850	.25127550	.84727450	-.07194150
1.74	-.02501200	.23883600	.85736400	-.07118800
1.75	-.02343750	.22656250	.86718750	-.07031250
1.76	-.02188800	.21446400	.87673600	-.06931200
1.77	-.02036650	.20254950	.88600050	-.06818350
1.78	-.01887600	.19082800	.89497200	-.06692400
1.79	-.01741950	.17930850	.90364150	-.06553050
1.80	-.01600000	.16800000	.91200000	-.06400000
1.81	-.01462050	.15691150	.92003850	-.06232950
1.82	-.01328400	.14605200	.92774800	-.06051600
1.83	-.01199350	.13543050	.93511950	-.05855650
1.84	-.01075200	.12505600	.94214400	-.05644800
1.85	-.00956250	.11493750	.94881250	-.05418750
1.86	-.00842800	.10508400	.95511600	-.05177200
1.87	-.00735150	.09550450	.96104550	-.04919850
1.88	-.00633600	.08620800	.96659200	-.04646400
1.89	-.00538450	.07720350	.97174650	-.04356550
1.90	-.00450000	.06850000	.97650000	-.04050000
1.91	-.00368550	.06010650	.98084350	-.03726450
1.92	-.00294400	.05203200	.98476800	-.03385600
1.93	-.00227850	.04428550	.98826450	-.03027150
1.94	-.00169200	.03687600	.99132400	-.02650800
1.95	-.00118750	.02981250	.99393750	-.02256250
1.96	-.00076800	.02310400	.99609600	-.01843200
1.97	-.00043650	.01675950	.99779050	-.01411350
1.98	-.00019600	.01078800	.99901200	-.00960400
1.99	-.00004950	.00519850	.99975150	-.00490050
2.00	0.00000000	-.00000000	1.00000000	0.00000000

TABLE A-2.— *Coefficients for four-point integration*

Z	C0	C1	C2	C3
1.00	-.00000000	-.00000000	0.00000000	.00000000
1.01	-.00002467	.00999917	.00002566	-.00000017
1.02	-.00009735	.01999339	.00010527	-.00000131
1.03	-.00021610	.02997780	.00024270	-.00000440
1.04	-.00037899	.03994763	.00044171	-.00001035
1.05	-.00058411	.04989818	.00070599	-.00002005
1.06	-.00082962	.05982486	.00103914	-.00003438
1.07	-.00111367	.06972317	.00144466	-.00005417
1.08	-.00143445	.07958869	.00192597	-.00008021
1.09	-.00179020	.08941710	.00248640	-.00011330
1.10	-.00217917	.09920417	.00312917	-.00015417
1.11	-.00259963	.10894574	.00385743	-.00020353
1.12	-.00304992	.11863776	.00467424	-.00026208
1.13	-.00352837	.12827627	.00558256	-.00033047
1.14	-.00403335	.13785739	.00658527	-.00040931
1.15	-.00456328	.14737734	.00768516	-.00049922
1.16	-.00511659	.15683243	.00888491	-.00060075
1.17	-.00569173	.16621904	.01018713	-.00071443
1.18	-.00628722	.17553366	.01159434	-.00084078
1.19	-.00690157	.18477287	.01310896	-.00098027
1.20	-.00753333	.19393333	.01473333	-.00113333
1.21	-.00818110	.20301180	.01646970	-.00130040
1.22	-.00884349	.21200513	.01832021	-.00148185
1.23	-.00951913	.22091024	.02028693	-.00167803
1.24	-.01020672	.22972416	.02237184	-.00188928
1.25	-.01090495	.23844401	.02457682	-.00211589
1.26	-.01161255	.24706699	.02690367	-.00235811
1.27	-.01232830	.25559040	.02935410	-.00261620
1.28	-.01305099	.26401163	.03192971	-.00289035
1.29	-.01377943	.27232814	.03463203	-.00318073
1.30	-.01451250	.28053750	.03746250	-.00348750
1.31	-.01524907	.28863737	.04042246	-.00381077
1.32	-.01598805	.29662549	.04351317	-.00415061
1.33	-.01672840	.30449970	.04673580	-.00450710
1.34	-.01746909	.31225793	.05009141	-.00488025

1.35	-.01820911	.31989818	.05358099	-.00527005
1.36	-.01894752	.32741856	.05720544	-.00567648
1.37	-.01968337	.33481727	.06096556	-.00609947
1.38	-.02041575	.34209259	.06486207	-.00653891
1.39	-.02114380	.34924290	.06889560	-.00699470
1.40	-.02186667	.35626667	.07306667	-.00746667
1.41	-.02258353	.36316244	.07737573	-.00795463
1.42	-.02329362	.36992886	.08182314	-.00845838
1.43	-.02399617	.37656467	.08640916	-.00897767
1.44	-.02469045	.38306869	.09113397	-.00951221
1.45	-.02537578	.38943984	.09599766	-.01006172
1.46	-.02605149	.39567713	.10100021	-.01062585
1.47	-.02671693	.40177964	.10614153	-.01120423
1.48	-.02737152	.40774656	.11142144	-.01179648
1.49	-.02801467	.41357717	.11683966	-.01240217
1.50	-.02864583	.41927083	.12239583	-.01302083
1.51	-.02926450	.42482700	.12808950	-.01365200
1.52	-.02987019	.43024523	.13392011	-.01429515
1.53	-.03046243	.43552514	.13988703	-.01494973
1.54	-.03104082	.44066646	.14598954	-.01561518
1.55	-.03160495	.44566901	.15222682	-.01629089
1.56	-.03215445	.45053269	.15859797	-.01697621
1.57	-.03268900	.45525750	.16510200	-.01767050
1.58	-.03320829	.45984353	.17173781	-.01837305
1.59	-.03371203	.46429094	.17850423	-.01908313
1.60	-.03420000	.46860000	.18540000	-.01980000
1.61	-.03467197	.47277107	.19242376	-.02052287
1.62	-.03512775	.47680459	.19957407	-.02125091
1.63	-.03556720	.48070110	.20684940	-.02198330
1.64	-.03599019	.48446123	.21424811	-.02271915
1.65	-.03639661	.48808568	.22176849	-.02345755
1.66	-.03678642	.49157526	.22940874	-.02419758
1.67	-.03715957	.49493087	.23716696	-.02493827
1.68	-.03751605	.49815349	.24504117	-.02567861
1.69	-.03785590	.50124420	.25302930	-.02641760

1.70	-.03817917	.50420417	.26112917	-.02715417
1.71	-.03848593	.50703464	.26933853	-.02788723
1.72	-.03877632	.50973696	.27765504	-.02861568
1.73	-.03905147	.51231257	.28607626	-.02933837
1.74	-.03930855	.51476299	.29459967	-.03005411
1.75	-.03955078	.51708984	.30322266	-.03076172
1.76	-.03977739	.51929483	.31194251	-.03145995
1.77	-.03998863	.52137974	.32075643	-.03214753
1.78	-.04018482	.52334646	.32966154	-.03282318
1.79	-.04036627	.52519697	.33865486	-.03348557
1.80	-.04053333	.52693333	.34773333	-.03413333
1.81	-.04068640	.52855770	.35689380	-.03476510
1.82	-.04082589	.53007233	.36613301	-.03537945
1.83	-.04095223	.53147954	.37544763	-.03597493
1.84	-.04106592	.53278176	.38483424	-.03655008
1.85	-.04116745	.53398151	.39428932	-.03710339
1.86	-.04125735	.53508139	.40380927	-.03763331
1.87	-.04133620	.53608410	.41339040	-.03813830
1.88	-.04140459	.53699243	.42302891	-.03861675
1.89	-.04146313	.53780924	.43272093	-.03906703
1.90	-.04151250	.53853750	.44246250	-.03948750
1.91	-.04155337	.53918027	.45224956	-.03987647
1.92	-.04158645	.53974069	.46207797	-.04023221
1.93	-.04161250	.54022200	.47194350	-.04055300
1.94	-.04163229	.54062753	.48184181	-.04083705
1.95	-.04164661	.54096068	.49176849	-.04108255
1.96	-.04165632	.54122496	.50171904	-.04128768
1.97	-.04166227	.54142397	.51168886	-.04145057
1.98	-.04166535	.54156139	.52167327	-.04156931
1.99	-.04166650	.54164100	.53166750	-.04164200
2.00	-.04166667	.54166667	.54166667	-.04166667

TABLE A-3. — *Coefficients for six-point interpolation*

Z	C0	C1	C2	C3	C4	C5
2.00	0.00000000	0.00000000	1.00000000	0.00000000	0.00000000	0.00000000
2.01	.00082880	-.00659840	.99987214	.00673603	-.00083885	.00000029
2.02	.00164709	-.01305409	.99947750	.01362118	-.00169393	.00000225
2.03	.00245281	-.01935826	.99880045	.02067012	-.00257259	.00000748
2.04	.00324403	-.02550272	.99782656	.02789632	-.00348160	.00001741
2.05	.00401895	-.03147988	.99654258	.03531211	-.00442715	.00003340
2.06	.00477586	-.03728275	.99493642	.04292866	-.00541487	.00005668
2.07	.00551320	-.04290490	.99299713	.05075603	-.00644985	.00008839
2.08	.00622950	-.04834048	.99071488	.05880320	-.00753664	.00012954
2.09	.00692343	-.05358418	.98808090	.06707807	-.00867927	.00018105
2.10	.00759375	-.05863125	.98508750	.07558750	-.00988125	.00024375
2.11	.00823933	-.06347744	.98172802	.08433735	-.01114561	.00031836
2.12	.00885914	-.06811904	.97799680	.09333248	-.01247488	.00040550
2.13	.00945226	-.07255283	.97388918	.10257678	-.01387113	.00050572
2.14	.01001788	-.07677607	.96940146	.11207322	-.01533595	.00061946
2.15	.01055527	-.08078652	.96453086	.12182383	-.01687051	.00074707
2.16	.01106381	-.08458240	.95927552	.13182976	-.01847552	.00088883
2.17	.01154295	-.08816237	.95363447	.14209130	-.02015128	.00104493
2.18	.01199225	-.09152553	.94760758	.15260790	-.02189769	.00121549
2.19	.01241136	-.09467142	.94119558	.16337819	-.02371423	.00140052
2.20	.01280000	-.09760000	.93440000	.17440000	-.02560000	.00160000
2.21	.01315798	-.10031161	.92722315	.18567042	-.02755374	.00181381
2.22	.01348519	-.10280699	.91966810	.19718578	-.02957383	.00204175
2.23	.01378159	-.10508727	.91173866	.20894170	-.03165829	.00228359
2.24	.01404723	-.10715392	.90343936	.22093312	-.03380480	.00253901

TABLE A-3. — *Coefficients for six-point interpolation* — Continued

2.25	.01428223	−.10900879	.89477539	.23315430	−.03601074	.00280762
2.26	.01448676	−.11065405	.88575262	.24559886	−.03827317	.00308898
2.27	.01466108	−.11209221	.87637755	.25825982	−.04058884	.00338261
2.28	.01480550	−.11332608	.86665728	.27112960	−.04295424	.00368794
2.29	.01492041	−.11435879	.85659951	.28420005	−.04536556	.00400437
2.30	.01500625	−.11519375	.84621250	.29746250	−.04781875	.00433125
2.31	.01506351	−.11583465	.83550503	.31090774	−.05030950	.00466788
2.32	.01509274	−.11628544	.82448640	.32452608	−.05283328	.00501350
2.33	.01509454	−.11655033	.81316640	.33830737	−.05538532	.00536734
2.34	.01506958	−.11663377	.80155526	.35224102	−.05796065	.00572856
2.35	.01501855	−.11654043	.78966367	.36631602	−.06055410	.00609629
2.36	.01494221	−.11627520	.77750272	.38052096	−.06316032	.00646963
2.37	.01484133	−.11584317	.76508388	.39484409	−.06577378	.00684765
2.38	.01471675	−.11524963	.75241898	.40927330	−.06838879	.00722939
2.39	.01456934	−.11450003	.73952019	.42379617	−.07099952	.00761384
2.40	.01440000	−.11360000	.72640000	.43840000	−.07360000	.00800000
2.41	.01420966	−.11255531	.71307116	.45307181	−.07618414	.00838682
2.42	.01399929	−.11137189	.69954670	.46779838	−.07874573	.00877325
2.43	.01376987	−.11005577	.68583988	.48256629	−.08127848	.00915821
2.44	.01352243	−.10861312	.67196416	.49736192	−.08377600	.00954061
2.45	.01325801	−.10705020	.65793320	.51217148	−.08623184	.00991934
2.46	.01297766	−.10537335	.64376082	.52698106	−.08863947	.01029328
2.47	.01268246	−.10358901	.62946096	.54177661	−.09099234	.01066132
2.48	.01237350	−.10170368	.61504768	.55654400	−.09328384	.01102234
2.49	.01205190	−.09972390	.60053513	.57126904	−.09550735	.01137519

TABLE A-3. — *Coefficients for six-point interpolation* — Continued

2.50	.01171875	-.09765625	.58593750	.58593750	-.09765625	.01171875
2.51	.01137519	-.09550735	.57126904	.60053513	-.09972390	.01205190
2.52	.01102234	-.09328384	.55654400	.61504768	-.10170368	.01237350
2.53	.01066132	-.09099234	.54177661	.62946096	-.10358901	.01268246
2.54	.01029328	-.08863947	.52698106	.64376082	-.10537335	.01297766
2.55	.00991934	-.08623184	.51217148	.65793320	-.10705020	.01325801
2.56	.00954061	-.08377600	.49736192	.67196416	-.10861312	.01352243
2.57	.00915821	-.08127848	.48256629	.68583988	-.11005577	.01376987
2.58	.00877325	-.07874573	.46779838	.69954670	-.11137189	.01399929
2.59	.00838682	-.07618414	.45307181	.71307116	-.11255531	.01420966
2.60	.00800000	-.07360000	.43840000	.72640000	-.11360000	.01440000
2.61	.00761384	-.07099952	.42379617	.73952019	-.11450003	.01456934
2.62	.00722939	-.06838879	.40927330	.75241898	-.11524963	.01471675
2.63	.00684765	-.06577378	.39484409	.76508388	-.11584317	.01484133
2.64	.00646963	-.06316032	.38052096	.77750272	-.11627520	.01494221
2.65	.00609629	-.06055410	.36631602	.78966367	-.11654043	.01501855
2.66	.00572856	-.05796065	.35224102	.80155526	-.11663377	.01506958
2.67	.00536734	-.05538532	.33830737	.81316640	-.11655033	.01509454
2.68	.00501350	-.05283328	.32452608	.82448640	-.11628544	.01509274
2.69	.00466788	-.05030950	.31090774	.83550503	-.11583465	.01506351
2.70	.00433125	-.04781875	.29746250	.84621250	-.11519375	.01500625
2.71	.00400437	-.04536556	.28420005	.85659951	-.11435879	.01492041
2.72	.00368794	-.04295424	.27112960	.86665728	-.11332608	.01480550
2.73	.00338261	-.04058884	.25825982	.87637755	-.11209221	.01466108
2.74	.00308898	-.03827317	.24559886	.88575262	-.11065405	.01448676

TABLE A-3. — *Coefficients for six-point interpolation* — Continued

2.75	.00280762	-.03601074	.23315430	.89477539	-.10900879	.01428223
2.76	.00253901	-.03380480	.22093312	.90343936	-.10715392	.01404723
2.77	.00228359	-.03165829	.20894170	.91173866	-.10508727	.01378159
2.78	.00204175	-.02957383	.19718578	.91966810	-.10280699	.01348519
2.79	.00181381	-.02755374	.18567042	.92722315	-.10031161	.01315798
2.80	.00160000	-.02560000	.17440000	.93440000	-.09760000	.01280000
2.81	.00140052	-.02371423	.16337819	.94119558	-.09467142	.01241136
2.82	.00121549	-.02189769	.15260790	.94760758	-.09152553	.01199225
2.83	.00104493	-.02015128	.14209130	.95363447	-.08816237	.01154295
2.84	.00088883	-.01847552	.13182976	.95927552	-.08458240	.01106381
2.85	.00074707	-.01687051	.12182383	.96453086	-.08078652	.01055527
2.86	.00061946	-.01533595	.11207322	.96940146	-.07677607	.01001788
2.87	.00050572	-.01387113	.10257678	.97388918	-.07255283	.00945226
2.88	.00040550	-.01247488	.09333248	.97799680	-.06811904	.00885914
2.89	.00031836	-.01114561	.08433735	.98172802	-.06347744	.00823933
2.90	.00024375	-.00988125	.07558750	.98508750	-.05863125	.00759375
2.91	.00018105	-.00867927	.06707807	.98808090	-.05358418	.00692343
2.92	.00012954	-.00753664	.05880320	.99071488	-.04834048	.00622950
2.93	.00008839	-.00644985	.05075603	.99299713	-.04290490	.00551320
2.94	.00005668	-.00541487	.04292866	.99493642	-.03728275	.00477586
2.95	.00003340	-.00442715	.03531211	.99654258	-.03147988	.00401895
2.96	.00001741	-.00348160	.02789632	.99782656	-.02550272	.00324403
2.97	.00000748	-.00257259	.02067012	.99880045	-.01935826	.00245281
2.98	.00000225	-.00169393	.01362118	.99947750	-.01305409	.00164709
2.99	.00000029	-.00083885	.00673603	.99987214	-.00659840	.00082880
3.00	0.00000000	0.00000000	0.00000000	1.00000000	0.00000000	0.00000000

TABLE A-4. — *Coefficients for six-point differentiation*

Z	C0	C1	C2	C3	C4	C5
2.00	.08333333	-.66666667	0.00000000	.66666667	-.08333333	0.00000000
2.01	.08238966	-.65285645	-.02585410	.68080444	-.08456905	.00008551
2.02	.08123383	-.63813450	-.05333367	.69646967	-.08656950	.00033417
2.03	.07987849	-.62256295	-.08231644	.71354210	-.08927555	.00073434
2.04	.07833600	-.60620267	-.11268267	.73190400	-.09262933	.00127467
2.05	.07661949	-.58911328	-.14431510	.75144010	-.09657422	.00194401
2.06	.07473783	-.57135317	-.17709900	.77203767	-.10105483	.00273150
2.07	.07270566	-.55297945	-.21092210	.79358644	-.10601705	.00362651
2.08	.07053333	-.53404800	-.24567467	.81597867	-.11140800	.00461867
2.09	.06823199	-.51461345	-.28124944	.83910910	-.11717605	.00569784
2.10	.06581250	-.49472917	-.31754167	.86287500	-.12327083	.00685417
2.11	.06328549	-.47444728	-.35444910	.88717610	-.12964322	.00807801
2.12	.06066133	-.45381867	-.39187200	.91191467	-.13624533	.00936000
2.13	.05795016	-.43289295	-.42971310	.93699544	-.14303055	.01069101
2.14	.05516183	-.41171850	-.46787767	.96232567	-.14995350	.01206217
2.15	.05230599	-.39034245	-.50627344	.98781510	-.15697005	.01346484
2.16	.04939200	-.36881067	-.54481067	1.01337600	-.16403733	.01489067
2.17	.04642899	-.34716778	-.58340210	1.03892310	-.17111372	.01633151
2.18	.04342583	-.32545717	-.62196300	1.06437367	-.17815883	.01777950
2.19	.04039116	-.30372095	-.66041110	1.08964744	-.18513355	.01922701
2.20	.03733333	-.28200000	-.69866667	1.11466667	-.19200000	.02066667
2.21	.03426049	-.26033395	-.73665244	1.13935610	-.19872155	.02209134
2.22	.03118050	-.23876117	-.77429367	1.16364300	-.20526283	.02349417
2.23	.02810099	-.21731878	-.81151810	1.18745710	-.21158972	.02486851
2.24	.02502933	-.19604267	-.84825600	1.21073067	-.21766933	.02620800

TABLE A-4. — *Coefficients for six-point differentiation* — Continued

2.25	.02197266	-.17496745	-.88444010	1.23339844	-.22347005	.02750651
2.26	.01893783	-.15412650	-.92000567	1.25539767	-.22896150	.02875817
2.27	.01593149	-.13355195	-.95489044	1.27666810	-.23411455	.02995734
2.28	.01296000	-.11327467	-.98903467	1.29715200	-.23890133	.03109867
2.29	.01002949	-.09332428	-1.02238110	1.31679410	-.24329522	.03217701
2.30	.00714583	-.07372917	-1.05487500	1.33554167	-.24727083	.03318750
2.31	.00431466	-.05451645	-1.08646410	1.35334444	-.25080405	.03412551
2.32	.00154133	-.03571200	-1.11709867	1.37015467	-.25387200	.03498667
2.33	-.00116901	-.01734045	-1.14673144	1.38592710	-.25645305	.03576684
2.34	-.00381150	.00057483	-1.17531767	1.40061900	-.25852683	.03646217
2.35	-.00638151	.01801172	-1.20281510	1.41419010	-.26007422	.03706901
2.36	-.00887467	.03494933	-1.22918400	1.42660267	-.26107733	.03758400
2.37	-.01128684	.05136805	-1.25438710	1.43782144	-.26151955	.03800401
2.38	-.01361417	.06724950	-1.27838967	1.44781367	-.26138550	.03832617
2.39	-.01585301	.08257655	-1.30115944	1.45654910	-.26066105	.03854784
2.40	-.01800000	.09733333	-1.32266667	1.46400000	-.25933333	.03866667
2.41	-.02005201	.11150522	-1.34288410	1.47014110	-.25739072	.03868051
2.42	-.02200617	.12507883	-1.36178700	1.47494967	-.25482283	.03858750
2.43	-.02385984	.13804205	-1.37935310	1.47840544	-.25162055	.03838601
2.44	-.02561067	.15038400	-1.39556267	1.48049067	-.24777600	.03807467
2.45	-.02725651	.16209505	-1.41039844	1.48119010	-.24328255	.03765234
2.46	-.02879550	.17316683	-1.42384567	1.48049100	-.23813483	.03711817
2.47	-.03022601	.18359222	-1.43589210	1.47838310	-.23232872	.03647151
2.48	-.03154667	.19336533	-1.44652800	1.47485867	-.22586133	.03571200
2.49	-.03275634	.20248155	-1.45574610	1.46991244	-.21873105	.03483951

TABLE A-4.— *Coefficients for six-point differentiation* — Continued

2.50	-.03385417	.21093750	-1.46354167	1.46354167	-.21093750	.03385417
2.51	-.03483951	.21873105	-1.46991244	1.45574610	-.20248155	.03275634
2.52	-.03571200	.22586133	-1.47485867	1.44652800	-.19336533	.03154667
2.53	-.03647151	.23232872	-1.47838310	1.43589210	-.18359222	.03022601
2.54	-.03711817	.23813483	-1.48049100	1.42384567	-.17316683	.02879550
2.55	-.03765234	.24328255	-1.48119010	1.41039844	-.16209505	.02725651
2.56	-.03807467	.24777600	-1.48049067	1.39556267	-.15038400	.02561067
2.57	-.03838601	.25162055	-1.47840544	1.37935310	-.13804205	.02385984
2.58	-.03858750	.25482283	-1.47494967	1.36178700	-.12507883	.02200617
2.59	-.03868951	.25739072	-1.47014110	1.34288410	-.11150522	.02005201
2.60	-.03866667	.25933333	-1.46400000	1.32266667	-.09733333	.01800000
2.61	-.03854784	.26066105	-1.45654910	1.30115944	-.08257655	.01585301
2.62	-.03832617	.26138550	-1.44781367	1.27838967	-.06724950	.01361417
2.63	-.03800401	.26151955	-1.43782144	1.25438710	-.05136805	.01128684
2.64	-.03758400	.26107733	-1.42660267	1.22918400	-.03494933	.00887467
2.65	-.03706901	.26007422	-1.41419010	1.20281510	-.01801172	.00638151
2.66	-.03646217	.25852683	-1.40061900	1.17531767	-.00057483	.00381150
2.67	-.03576684	.25645305	-1.38592710	1.14673144	.01734045	.00116901
2.68	-.03498667	.25387200	-1.37015467	1.11709867	.03571200	-.00154133
2.69	-.03412551	.25080405	-1.35334444	1.08646410	.05451645	-.00431466
2.70	-.03318750	.24727083	-1.33554167	1.05487500	.07372917	-.00714583
2.71	-.03217701	.24329522	-1.31679410	1.02238110	.09332428	-.01002949
2.72	-.03109867	.23890133	-1.29715200	.98903467	.11327467	-.01296000
2.73	-.02995734	.23411455	-1.27666810	.95489044	.13355195	-.01593149
2.74	-.02875817	.22896150	-1.25539767	.92000567	.15412650	-.01893783

TABLE A-4. — *Coefficients for six-point differentiation* — Continued

2.75	-.02750651	.22347005	-1.23339844	.88444010	.17496745	-.02197266
2.76	-.02620300	.21766933	-1.21073067	.84825600	.19604267	-.02502933
2.77	-.02486851	.21158972	-1.18745710	.81151810	.21731878	-.02810099
2.78	-.02349417	.20526283	-1.16364300	.77429367	.23876117	-.03118050
2.79	-.02209134	.19872155	-1.13935610	.73665244	.26033395	-.03426049
2.80	-.02066667	.19200000	-1.11466667	.69866667	.28200000	-.03733333
2.81	-.01922701	.18513355	-1.08964744	.66041110	.30372095	-.04039116
2.82	-.01777950	.17815883	-1.06437367	.62196300	.32545717	-.04342583
2.83	-.01633151	.17111372	-1.03892310	.58340210	.34716778	-.04642899
2.84	-.01489067	.16403733	-1.01337600	.54481067	.36881067	-.04939200
2.85	-.01346484	.15697005	-.98781510	.50627344	.39034245	-.05230599
2.86	-.01206217	.14995350	-.96232567	.46787767	.41171850	-.05516183
2.87	-.01069191	.14303055	-.93699544	.42971310	.43289295	-.05795016
2.88	-.00936000	.13624533	-.91191467	.39187200	.45381867	-.06066133
2.89	-.00807801	.12964322	-.88717610	.35444910	.47444728	-.06328549
2.90	-.00685417	.12327083	-.86287500	.31754167	.49472917	-.06581250
2.91	-.00569784	.11717605	-.83910910	.28124944	.51461345	-.06823199
2.92	-.00461867	.11140800	-.81597867	.24567467	.53404800	-.07053333
2.93	-.00362651	.10601705	-.79358644	.21092210	.55297945	-.07270566
2.94	-.00273150	.10105483	-.77203767	.17709900	.57135317	-.07473783
2.95	-.00194401	.09657422	-.75144010	.14431510	.58911328	-.07661849
2.96	-.00127467	.09262933	-.73190400	.11268267	.60620267	-.07833600
2.97	-.00073434	.08927555	-.71354210	.08231644	.62256295	-.07987849
2.98	-.00033417	.08656950	-.69646967	.05333367	.63813450	-.08123383
2.99	-.00008551	.08456905	-.68080444	.02585410	.65285645	-.08238966
3.00	0.00000000	.08333333	-.66666667	0.00000000	.66666667	-.08333333

EXAMPLES USING TABLES A-1 AND A-2

xample 1.—Interpolate a value of Y at $x=0.07$

ven the table of base points

x	$BY?$
-0.05	1.00
.00	1.15
.05	1.25
.10	1.25
.15	1.10
.20	.75
.25	.15

$$z=1+(0.07-0.05)/(0.10-0.05)=1.4$$

om table A-1:

$(C0= -0.072)\times1.15=$	-0.0828
$(C1= .696)\times1.25=$	$.8700$
$(C2= .424)\times1.25=$	$.5300$
$(C3= -.048)\times1.10=$	$-.0518$
$y_{0.07}=$	1.2644

ample 2.—Compute the integral from $x=0.00$ $x=0.07$

om 0.00 to 0.05

om table A-2:

$(C0=-0.04167)\times1.00=$	-0.0417
$(C1= .54167)\times1.15=$	$.6229$
$(C2= .54167)\times1.25=$	$.6771$
$(C3= -.04167)\times1.25=$	$-.0521$
	1.2062

om 0.05 to 0.07

om table A-2:

$(C0=-0.02187)\times1.15=$	-0.0252
$(C1= .35627)\times1.25=$	$.4453$
$(C2= .07307)\times1.25=$	$.0913$
$(C3= -.00747)\times1.10=$	$-.0082$
	$.5032$

$$\int_{0.00}^{0.07} y\,dx=(1.2062+0.5032)\times0.05=0.08547$$

EXAMPLES USING TABLES A-3 AND A-4

Example 3.—Given the same base points as in example 1 interpolate at $x=0.07$.

From table A-3:

$(C0=0.0144)\times1.00=$	0.0144
$(C1=-.1136)\times1.15=$	$-.1306$
$(C2= .7264)\times1.25=$	$.9080$
$(C3= .4384)\times1.25=$	$.5480$
$(C4=-.0736)\times1.10=$	$-.0810$
$(C5= .0080)\times .75=$	$.0060$
$y_{0.07}=$	1.2648

Example 4.—Interpolate for the differential of y at $x=0.07$

From table A-4:

$(C0=0.01800)\times1.00=$	0.0180
$(C1= .09733)\times1.15=$	$.1119$
$(C2=1.32267)\times1.25=$	1.6533
$(C3=1.46400)\times1.25=$	1.8300
$(C4= .25933)\times1.10=$	$-.2853$
$(C5= .03867)\times .75=$	$.0290$
	$.0143$

$$(dy/dx)_{0.07}=(0.0143/0.05)=0.286$$

NOTES FOR PROGRAMS A-1, A-2, A-3, AND A-4

TAB(I)	Input of tabular values, or base points, on which interpolation is based.
YINT(I)	The interpolated values.
YAINT(I)	Interpolated integral.
YPINT(I)	Interpolated derivative.
INTFOR( )	Subroutine for four-point interpolation.
INTSIX( )	Subroutine for six-point interpolation.
C(I)	Coefficients of equations 37 and 43.
ICOEF(I)	The matrices of equations 37 and 43.
YAO(I)	Coefficients of equation 42.
XI(I)	The index z of tables A-1, A-2, A-3, and A-4.

PROGRAM A-1.—Four-point interpolation and integration

```
      DIMENSION YINT(200),YAINT(200)
      COMMON/XX/TAB(100)
      READ(5,1000)(TAB(I),I=3,22)
 1000 FORMAT(20F4.2)
      TAB(1)=0.1
      TAB(2)=0.1
      X=0.3
      MS=-1
      DO 1001 I=1,151
      X=X+0.01
      CALL INTFOR(X,Y,YA,MS)
      YINT(I)=Y
      YAINT(I)=YA
 1001 CONTINUE
      WRITE(6,1002)(I,YINT(I),I=1,151)
 1002 FORMAT(* INTERPOLATED FUNCTION*/(8(I5,F10.4)))
      WRITE(6,1003) (I,YAINT(I),I=1,151)
 1003 FORMAT(* INTERPOLATED INTEGRAL*/(8(I5,F10.4)))
      STOP
      END
```

```
      SUBROUTINE INTFOR(X,Y,YA,MS)
      DIMENSION C(4),ICOEF(4,4)
      COMMON/XX/TAB(100)
      DATA ICOEF/4,-8,5,-1,-6,19,-14,3,6,-16,13,-3,-2,5,-4,1/
      AX=((X-TAB(1))/TAB(2))
      M=INT(AX)
      FX=AX-FLOAT(M)+1.0
      IF(M.EQ.MS) GO TO 102
      DO 100 I=1,4
      C(I)=0.0
      DO 100 J=1,4
  100 C(I)=C(I)+ICOEF(I,J)*TAB(M+J+1)
      CA=C(1)+C(2)/2.0+C(3)/3.0+C(4)/4.0
      MS=M
  102 Y=C(4)
      YA=C(4)/4.0
      DO 101 K=1,3
      Y=Y*FX+C(4-K)
  101 YA=YA*FX+C(4-K)/(4-K)
      Y=Y/2.0
      YA=(YA*FX-CA)*TAB(2)/2.0
      RETURN
      END
```

PROGRAM A-2.—Generation of four-point coefficients

```
      DIMENSION YINT(101,4),YAINT(101,4),ICOEF(4,4),XI(101), YAO(4)
      DATA ICOEF/4,-8,5,-1,-6,19,-14,3,6,-16,13,-3,-2,5,-4,1/
      YAO(1)=-17.0/24.0
      YAO(2)=5.0/24.0
      YAO(3)=-19.0/24.0
      YAO(4)=7.0/24.0
      DO 1000 J=1,101
      X=FLOAT(J)/100.0+0.99
      XI(J)=X
      DO 1001 I=1,4
      YINT(J,I)=ICOEF(4,I)
      DO 1002 K=1,3
 1002 YINT(J,I)=YINT(J,I)*X+ICOEF(4-K,I)
      YINT(J,I)=YINT(J,I)/2.0
      YAINT(J,I)=ICOEF(4,I)/4.0
      DO 1003 K=1,3
 1003 YAINT(J,I)=(YAINT(J,I)*X)+(FLOAT(ICOEF(4-K,I))/(4-K))
      YAINT(J,I)=(YAINT(J,I)/2.0)*X+YAO(I)
 1001 CONTINUE
 1000 CONTINUE
      DO 1005 J=1,101
      IF(J.EQ.1.OR.J.EQ.51) GO TO 1006
      GO TO 1007
 1006 WRITE(6,1008)
 1008 FORMAT(*1*)
 1007 IF(MOD(J,5).EQ.1) GO TO 1009
      GO TO 1010
 1009 WRITE(6,1011)
 1011 FORMAT(*0*)
 1010 WRITE(6,1012)(XI(J),(YINT(J,I),I=1,4))
 1012 FORMAT(* *,(35X,F5.2,4F12.8))
 1005 CONTINUE
      DO 1020 J=1,101
      IF(J.EQ.1.OR.J.EQ.51) GO TO 1013
      GO TO 1014
 1013 WRITE(6,1008)
 1014 IF(MOD(J,5).EQ.1) GO TO 1016
      GO TO 1017
 1016 WRITE(6,1011)
 1017 WRITE(6,1012)(XI(J),(YAINT(J,I),I=1,4))
 1020 CONTINUE
      STOP
      END
```

PROGRAM A-3.—Six-point interpolation and differentiation

```
      DIMENSION YINT(200),YPINT(200)
      COMMON/XX/TAB(100)
      READ(5,1000)(TAB(I),I=3,22)
 1000 FORMAT(20F4.2)
      TAB(1)=0.1
      TAB(2)=0.1
      X=0.3
      MS=-1
      DO 1001 I=1,151
      X=X+0.01
      CALL INTSIX(X,Y,YP,MS)
      YINT(I)=Y
      YPINT(I)=YP
 1001 CONTINUE
      WRITE(6,1002)(I,YINT(I),I=1,151)
 1002 FORMAT(* INTERPOLATED FUNCTION*/(8(I5,F10.4)))
      WRITE(6,1003)(I,YPINT(I),I=1,151)
 1003 FORMAT(* INTERPOLATED DERIVATIVE*/(8(I5,F10.4)))
      STOP
      END


      SUBROUTINE INTSIX(X,Y,YP,MS)
      DIMENSION  C(6),ICOEF(6,6)
      COMMON/XX/TAB(100)
      DATA ICOEF/432,-918,765,-313,63,-5,-2040,4436,-3754,1551,-314,25,
     1+080,-8752,7414,-3078,626,-50,-4080,8712,-7356,3058,-624,50,2040,
     2-4346,3661,-1521,311,-25,-408,868,-730,303,-62,5/
      AX=((X-TAB(1))/TAB(2))
      M=INT(AX)
      FX=AX-FLOAT(M)+2.0
      IF(M.EQ.MS) GO TO 102
      DO 100 I=1,6
      C(I)=0.0
      DO 100 J=1,6
  100 C(I)=C(I)+ICOEF(I,J)*TAB(M+J)
      MS=M
  102 Y=C(6)*FX+C(5)
      YP=5.0*C(6)
      DO 101 K=1,4
      Y=Y*FX+C(5-K)
  101 YP=YP*FX+(5-K)*C(6-K)
      Y=Y/24.0
      YP=YP/24.0/TAB(2)
      RETURN
      END
```

PROGRAM A-4.—Generation of six-point coefficients

```
      DIMENSION YINT(101,6),YPINT(101,6),ICOEF(6,6),XI(101)
      DATA ICOEF/432,-918,765,-313,63,-5,-2040,4436,-3754,1551,-314,25,
     14080,-8752,7414,-3078,626,-50,-4080,8712,-7356,3058,-624,50,2040,
     2-4346,3661,-1521,311,-25,-408,868,-730,303,-62,5/
      DO 1000 J=1,101
      X=FLOAT(J)/100.0+1.99
      XI(J)=X
      DO 1001 I=1,6
      YINT(J,I)=ICOEF(6,I)
      DO 1002 K=1,5
 1002 YINT(J,I)=YINT(J,I)*X+ICOEF(6-K,I)
      YINT(J,I)=YINT(J,I)/24.0
      YPINT(J,I)=ICOEF(6,I)*5.0
      DO 1003 K=1,4
 1003 YPINT(J,I)=YPINT(J,I)*X+ICOEF(6-K,I)*(5-K)
      YPINT(J,I)=YPINT(J,I)/24.0
 1001 CONTINUE
 1000 CONTINUE
      DO 1005 J=1,101
      IF(J.EQ.1.OR.J.EQ.51) GO TO 1006
      GO TO 1007
 1006 WRITE(6,1008)
 1008 FORMAT(*1*)
 1007 IF(MOD(J,5).EQ.1) GO TO 1009
      GO TO 1010
 1009 WRITE(6,1011)
 1011 FORMAT(*0*)
 1010 WRITE(6,1012)(XI(J),(YINT(J,I),I=1,6))
 1012 FORMAT(* *,(35X,F5.2,6F12.8))
 1005 CONTINUE
      DO 1020 J=1,101
      IF(J.EQ.1.OR.J.EQ.51) GO TO 1013
      GO TO 1014
 1013 WRITE(6,1008)
 1014 IF(MOD(J,5).EQ.1) GO TO 1016
      GO TO 1017
 1016 WRITE(6,1011)
 1017 WRITE(6,1012)(XI(J),(YPINT(J,I),I=1,6))
 1020 CONTINUE
      STOP
      END
```

APPENDIX B.—DATA ARRANGEMENT AND COMPUTER PROGRAMS FOR SMOOTHING

TABLE B-1. — *Data arrangement for two-point smoothing*

Measured data points		Base-point positions in x	Scale variate	Values of coefficients[1] for base points				
x	y		z	$YB1$	$YB2$	$YB3$	$YB4$	y
		0.4						
0 412	1.50		0.06	0.94	0.06			1.50
416	1.69		.08	.92	.08			1.69
458	2.02		.29	71	.29			2.02
.450	2 20		25	.75	.25			2.20
.464	2.50		.32	.68	.32			2.50
.500	2.50		.50	.50	.50			2.50
500	2.70		.50	.50	.50			2.70
.514	2.89		.57	.42	.57			2.89
.566	3 25		.83	.17	.83			3.25
.574	3 58		.87	.13	.87			3.58
.600	3.62	6	1.00		1 00			3.62
.624	3 66		.12		.88	0.12		3.66
.626	3.85		13		.87	.13		3.85
.660	3.55		.30		.70	.30		3.55
660	4.04		.30		70	.30		4.04
.686	3 88		.43		.57	43		3.88
.698	3.60		.49		.51	.49		3.60
.716	3.98		58		.42	.58		3.98
750	3.70		75		.25	.75		3.70
774	3.87		.87		.13	.87		3.87
		.8						
.806	3 58		.03			.97	.03	3.58
.844	3.69		.22			.78	22	3.69
.860	3 29		30			.70	.30	3.29
.866	3.56		.33			.67	.33	3 56
.886	3 27		.43			.57	.43	3.27
.910	3.10		.55			.45	.55	3.10
.914	3 23		.57			.43	.57	3.23
.956	2.74		.78			.22	78	2.74
.960	2.98		.80			.20	.80	2.98
.986	2 55		.93			.07	.93	2.55
		1.0						
				X_1	X_2	X_3	X_4	Y
				Conventional regression symbols				

[1] Equations 12 and 13.

TABLE B-2. —*Data arrangement for four-point smoothing*

Base-point positions in x	Scale variate z	Values of coefficients[1] for base points						
		*YB*1	*YB*2	*YB*3	*YB*4	*YB*5	*YB*6	*Y*
0.4								
.6								
	1.06	-0.0265	0.9913	0.0369	-0.0017			1.50
	1.08	-.0339	.9848	.0520	-.0029			1.69
	1.29	-.0731	.8263	.2766	-.0299			2.02
	1.25	-.0703	8672	.2266	-.0234			2 20
	1.32	-.0740	.7932	.3156	-.0348			2.50
	1.50	-.0625	5625	.5625	-.0625			2.50
	1.50	-.0625	5625	.5625	-.0625			2.70
	1.57	-.0527	.4655	.6570	-.0699			2.89
	1.83	-.0120	.1354	.9351	-.0586			3.25
	1 87	- 0074	.0955	.9610	- 0492			3.58
.8	2.00	.0000	.0000	1.0000	.0000	0.0000		3.62
	1.12		-.0465	.9666	.0862	-.0063		3.66
	1.13		-.0492	.9610	.0955	-.0074		3.85
	1.30		-.0735	.8155	.2895	-.0315		3.55
	1.30		- 0735	.8155	.2895	- 0315		4.04
	1.43		-.0699	.6570	.4655	-.0527		3.88
	1.49		-.0637	.5762	.5487	-.0612		3.60
	1.58		-.0512	.4517	.6701	-.0706		3.98
	1.75		-.0234	.2266	.8672	-.0703		3.70
	1.87		-.0074	.0955	.9610	-.0492		3.87
1.0								
	1.03		.0000	-.0141	.9978	0168	-.0004	3.58
	1.22			-.0669	.8950	.1908	-.0189	3.69
	1.30			-.0735	.8155	.2895	-.0315	3.29
	1.33			-.0741	.7417	.3289	-.0365	3.56
	1.43			-.0699	.6570	.4655	-.0527	3.27
	1.55			-.0557	.4933	.6304	-.0681	3.10
	1.57			-.0527	.4655	.6570	-.0699	3.23
	1.78			-.0189	.1908	.8950	-.0669	2.74
	1.80			-.0160	.1680	.9126	-.0640	2.98
	1.93			-.0023	.0443	.9883	-.0303	2.55
1.2								
1.4								
		X_1	X_2	X_3	X_4	X_5	X_6	Y
		Conventional regression symbols						

[1] From table A-1.

TABLE B-3. —*Data arrangement for six-point smoothing*

Base point positions in r	Scale variate z	Values of coefficients[1] for base points								
		$YB1$	$YB2$	$YB3$	$YB4$	$YB5$	$YB6$	$YB7$	$YB8$	Y
0.4										
.6										
.8										
	2.06	0.0048	-0.0373	0.9949	0.0429	-0.0054	0.0001			1.50
	2.08	.0062	-.0483	.9907	.0588	-.0075	.0001			1.69
	2.29	.0149	-.1144	.8566	.2842	-.0454	.0040			2.02
	2.25	.0143	-.1090	.8948	.2332	-.0360	.0028			2.20
	2.32	.0151	-.1163	.8245	.3245	-.0528	.0050			2.50
	2.50	.0117	-.0977	.5859	.5859	-.0977	.0117			2.50
	2.50	.0117	-.0977	.5859	.5859	-.0977	.0117			2.70
	2.57	.0092	-.0813	.4826	.6858	-.1101	.0138			2.89
	2.83	.0010	-.0202	.1421	.9536	-.0882	.0115			3.25
	2.87	.0005	-.0139	.1026	.9739	-.0726	.0095			3.58
1.0	3.00	.0000	.0000	.0000	1.0000	.0000	.0000			3.62
	2.12		.0089	-.0681	.9780	.0933	-.0125	0.0004		3.66
	2.13		.0095	-.0726	.9739	.1026	-.0139	.0005		3.85
	2.30		.0150	-.1152	.8462	.2975	-.0478	.0043		3.55
	2.30		.0150	-.1152	.8462	.2975	-.0478	.0043		4.04
	2.43		.0138	-.1101	.6858	.4826	-.0813	.0092		3.88
	2.49		.0121	-.0997	.6005	.5713	-.0955	.0114		3.60
	2.58		.0088	-.0787	.4678	.6995	-.1114	.0140		3.98
	2.75		.0028	-.0360	.2332	.8948	-.1090	.0143		3.70
	2.87		.0005	-.0139	.1026	.9739	-.0726	.0095		3.87
1.2										
	2.03			.0025	-.0194	.9988	.0207	-.0026	0.0000	3.58
	2.22			.0135	-.1028	.9197	.1972	-.0296	.0020	3.69
	2.30			.0150	-.1152	.8462	.2975	-.0478	.0043	3.29
	2.33			.0151	-.1166	.8132	.3383	-.0554	.0054	3.56
	2.43			.0138	-.1101	.6858	.4826	-.0813	.0092	3.27
	2.55			.0099	-.0862	.5122	.6579	-.1071	.0133	3.10
	2.57			.0092	-.0813	.4826	.6858	-.1101	.0138	3.23
	2.78			.0020	-.0296	.1972	.9197	-.1028	.0135	2.74
	2.80			.0016	-.0256	.1744	.9344	-.0976	.0128	2.98
	2.93			.0001	-.0064	.0508	.9930	-.0429	.0055	2.55
1.4										
1.6										
1.8										
		X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	Y
		Conventional regression symbols								

[1]From table A-3

NOTES FOR PROGRAMS B-1 AND B-2

X(I)	Input conventional regression data from tables B-2 and B-3.
SSX(I)	Sums of squares and products.
AA(I)	Matrix to be inverted.
MINV(AA,N,D,L,M)	System library routine for matrix inversion.
B(I)	Regression coefficients or base points.
INTFOR(V,Y,YA,MS)	See program A-1.
INTSIX(V,Y,YP,MS)	See program A-3.

```
      DIMENSION  X(30,10),SSX(10,10),L(10),M(10),B(10),YINT(60),YAINT(60)
     1,AA(6,6)
      COMMON/XX/ TAB(100)
      READ(5,1000)((X(I,J),J=1,7),I=1,30)
 1000 FORMAT(7F5.4)
      DO 1002 I=1,10
      DO 1002 J=1,10
 1002 SSX(I,J)=0.0
      DO 1003 I=1,7
      DO 1004 J=I,7
      DO 1005 K=1,30
 1005 SSX(I,J)=SSX(I,J)+X(K,I)*X(K,J)
 1004 CONTINUE
 1003 CONTINUE
      DO 1006 I=1,7
      DO 1006 J=I,7
 1006 SSX(J,I)=SSX(I,J)
      WRITE(6,1010)((SSX(I,J),I=1,7),J=1,7)
 1010 FORMAT(* *,(7F12.8))
      DO 1020 I=1,6
      DO 1020 J=1,6
 1020 AA(I,J)=SSX(I,J)
      CALL MINV(AA,6,D,L,M)
      WRITE(6,1011)((AA(I,J),I=1,6),J=1,6)
 1011 FORMAT(* *,(6F13.8))
      DO 1007 I=1,6
      B(I)=0.0
      DO 1008 J=1,6
 1008 B(I)=B(I)+AA(I,J)*SSX(J,7)
 1007 CONTINUE
      WRITE(6,1011)(B(I),I=1,6)
      TAB(1)=0.4
      TAB(2)=0.2
      DO 1012 I=3,8
 1012 TAB(I)=B(I-2)
      V=0.6
      MS=-1
      DO 1013 I=1,31
      CALL INTFOR(V,Y,YA,MS)
      YINT(I)=Y
      YAINT(I)=YA
      V=V+0.12
 1013 CONTINUE
      YAINT(1)=0.0
      WRITE(6,1014)(I,YINT(I),I=1,31)
 1014 FORMAT(* INTERPOLATED FUNCTION*/(8(I5,F10.6)))
      WRITE(6,1015)(I,YAINT(I),I=1,31)
 1015 FORMAT(* INTERPOLATED INTEGRAL*/(8(I5,F10.6)))
      STOP
      END
```

PROGRAM B-2.—Six-point smoothing

```
      DIMENSION X(30,10),SSX(10,10),L(10),M(10),B(10),YINT(60),YPINT(60)
     1,AA(8,8)
      COMMON/XX/ TAB(100)
      READ(5,1000)((X(I,J),J=1,9),I=1,30)
 1000 FORMAT(9F5.4)
      DO 1002 I=1,10
      DO 1002 J=1,10
 1002 SSX(I,J)=0.0
      DO 1003 I=1,9
      DO 1004 J=I,9
      DO 1005 K=1,30
 1005 SSX(I,J)=SSX(I,J)+X(K,I)*X(K,J)
 1004 CONTINUE
 1003 CONTINUE
      DO 1006 I=1,9
      DO 1006 J=I,9
 1006 SSX(J,I)=SSX(I,J)
      WRITE(6,1010)((SSX(I,J),I=1,9),J=1,9)
 1010 FORMAT(* *,(9F12.8))
      DO 1020 I=1,8
      DO 1020 J=1,8
 1020 AA(I,J)=SSX(I,J)
      CALL MINV(AA,8,D,L,M)
      WRITE(6,1011)((AA(I,J),I=1,8),J=1,8)
 1011 FORMAT(* *,(8F13.8))
      DO  1007 I=1,8
      B(I)=0.0
      DO 1008 J=1,8
 1008 B(I)=B(I)+AA(I,J)*SSX(J,9)
 1007 CONTINUE
      WRITE(6,1011)(B(I),I=1,8)
      TAB(1)=0.4
      TAB(2)=0.2
      DO 1012 I=3,10
 1012 TAB(I)=B(I-2)
      V=0.8
      MS=-1
      DO 1013 I=1,31
      CALL INTSIX(V,Y,YP,MS)
      YINT(I)=Y
      YPINT(I)=YP
      V=V+0.02
 1013 CONTINUE
      WRITE(6,1014)(I,YINT(I),I=1,31)
 1014 FORMAT(* INTERPOLATED FUNCTION*/(8(I5,F10.6)))
      WRITE(6,1015)(I,YPINT(I),I=1,31)
 1015 FORMAT(*  INTERPOLATED DERIVATIVE*/(8(I5,F10.6)))
      STOP
      END
```

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